1.0

1.1

1.25 1.4 1.6

2.8 2.5

3 2 2.2

3 6

4 0 2.0

1.8

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DISTRIBUTIONS ON THE CIRCLE AND SPHERE,

by

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ABSTRACT

A survey is made of the mathematical properties of, and the
arithmetic relationships between various distributions on the
circle and the sphere. Some new results are given. The Brownian
motion and Angular Gaussian distributions are shown in computer
drawn graphs to bracket the von Mises-Fisher distributions.

1. Introduction

Because I began my work on directional data while in Professor Moran's Department, it seems appropriate to take up the topic again for this volume in his honour. My motivation then was palaeomagnetism which has since greatly grown in scientific importance. The original direction of magnetization of rocks proved to be an invaluable clue to the discovery of plate tectonics. Later, discussions with ornithologists about the navigational and homing ability of birds led me to an interest in directions in two dimensions. In the intervening 25 years, many statisticians have created a sizeable literature, and a book (Mardia, 1972), on the statistics of directions. Though the theory often applies to an arbitrary number of dimensions, we will consider only 2 and 3 dimensions.

Almost all the parametric work has been based on one distributional type, introduced in another context by von Mises (1918) for the circle and developed for palaeomagnetism by Fisher (1953).

The nomenclature is unfortunate because Arnold (1941) in an unpublished M.I.T. Ph.D. Thesis derived the distribution named after Fisher and gave the maximum likelihood estimation procedures for both the von Mises and Fisher distributions. He also derived the Brownian motion densities (6) and (7).

The von Mises density is

$$vm(\theta,\kappa) = \frac{1}{2\pi I_0(\kappa)} \exp\left(\kappa\cos\theta\right) \tag{1}$$

where $\kappa \geq 0$ and $I_0(\kappa)$ is a Bessel function. The observed direction r makes an angle θ with the "mean" direction μ so $r \cdot \mu = \cos\theta$.

The Fisher density is, with $\kappa \geq 0$,

$$fi(\theta, \phi, \kappa = \frac{\kappa}{4\pi \sinh \kappa} \exp(\kappa \cos \theta) \tag{2}$$

using spherical polar coordinates (θ, ϕ) so that $r \cdot \mu = \cos \theta$. In both cases, κ is a "concentration" parameter. When $\kappa = 0$, the density is constant. When κ is large, the probability is concentrated about $\theta = 0$. The exponential form makes inference procedures easy to develop. It is symmetric about the mean, or polar, direction. It shares some statistical characterizations of the Gaussian distribution, which it is, approximately, for large κ. See e.g. Mardia (1975). More important, the exponential form of (1) and (2) makes statistical inference procedures fairly easy to develop.

Multipolar forms may be created by replacing θ and by $p\theta$, p an integer. When p=2, the density is proportional to

$$\exp(\kappa \cos^2 \theta) \tag{3}$$

and gives bipolar ($\kappa > 0$) and girdle distributions ($\kappa < 0$) on the sphere. Its generalization has density proportional to

$$\exp(\kappa_1 x_1^2 + \kappa_2 x_2^2 + \kappa_3 x_3^2) \tag{4}$$

where $x = (x_1, x_2, x_3)$ is the observed unit vector referred to three orthogonal axes. If the data is $x^{(1)}, \ldots x^{(n)}$, then the statistics of (3) and (4) depend upon the eigen values and vectors of $\sum_1^n x^{(i)} x^{(i)'}$, whereas the statistics of (1) and (2) are based on $\sum_1^n x^{(i)}$. Here x is a column vector, x' its transpose.

Fisher pointed out that the distributions (1) and (2) can be

obtained from a multivariate Gaussian by conditioning as will be explained in the beginning of §2. The distributions (3) and (4) also come from a Fisherian construction. Let X_1, X_2, X_3 have the joint density

$$(2\pi)^{-3/2}(\sigma_1 \ \sigma_2 \ \sigma_3)^{-1/2}\exp{-1/2}(x_1^2/\sigma_1^2 + x_2^2/\sigma_2^2 + x_3^2/\sigma_3^2)$$

If $R^2 = X_1^2 + X_2^2 + X_3^2$ then the distribution of the direction of X when R is conditioned to be unity is the Bingham distribution. If $\sigma_1 = \sigma_2$, this reduces to (3). If $\sigma_1 = \sigma_2 >> \sigma_3$, it clearly makes (3) a girdle distribution with the probability concentrated around a great circle in the x_1-x_2 plane.

There has always been the possibility of constructing densities which are the exponentials of higher order polynomials in the components of x. It is natural to rearrange these to obtain

$$\exp \Sigma a_p S_p(\theta, \phi)$$

where $S_m(\theta, \phi)$ are surface harmonics and θ & ϕ are spherical polar angles. Recently Beran (1979) has suggested how, despite the complicated norming factor, to find maximum likelihood estimates of the a_p's. Our efforts to apply his method will be reported in Watson (1981). For p>2, these distributions cannot have a Gaussian origin. It is not known whether they could arise from diffusions of the types mentioned below.

On the unit circle, one may get distributions by "rolling up" distributions on the line. This was a topic of interest to

mathematicians in the 30's and 40's — see e.g. Levy (1939),
Haviland (1941), Wintner (1947). Arnold (1941) in his thesis
considered this process and diffusion and suggested (1), (2), (3)
and (6). Thus a density $f(x)$ on the line becomes, on the unit
circle,

$$h(x) = \sum_{-\infty}^{\infty} f(x + 2\pi p). \tag{5}$$

If the Gaussian is used, this procedure leads to the Brownian
density (6) mentioned in the next paragraph. The procedure has
no analogue on the sphere since it is not a developable surface. It is
an interesting conjecture that distributions on ruled surfaces
will, one day, be generated from linear densities.

   Another simple process for generating circular from linear
or planar distributions is to use bi-linear (or linear fractional)
transformations. This suggests e.g. putting $x = \tan \theta/2$ which
sends the standard cauchy into the uniform distribution. There
are also some connections with number theory — see e.g. Stapleton
(1963) and Schmidt (1977). We will discuss these on another
occasion. We turn now to the major other source of distributions —
diffusion processes.

   If a particle starts at $\theta = 0$ and executes symmetrical
Brownian motion on the circumference of a unit circle with a
variance σ^2 per unit time, the density at θ at time t is

$$br_2(\theta, V) = \sum_{p = -\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi t}} \exp\left\{-\frac{(\theta - 2\pi p)^2}{2\sigma^2 t}\right\}$$

$$= \frac{1}{2\pi}\left\{1 + 2\sum_{1}^{\infty} \exp\left(-\frac{p^2 V}{2}\right)\cos(p\,\theta)\right\} \tag{6}$$

where $V = \sigma^2 t$. On the line the density would be Gaussian with
mean zero and variance V; this density when substituted in (5)
gives an alternative form of (6). A symmetrical Brownian motion,
starting at the $\theta = 0$ point of a spherical coordinate system,
yields a density on a sphere of unit radius given by

$$br_3(\theta,\phi,v) = \frac{1}{4\pi} \sum_{p=0}^{\infty} (2p + 1) \exp(-p(p+1)V/4) \; P_p(\cos\theta) \quad (7)$$

where $P_p(\cos\theta)$ is the pth Legendre polynomial in $\cos\theta$. These
distributions are the probabilistic, rather than statistical,
Gaussian analogues of the circle and sphere. In a later section
we will see other diffusion processes which lead to the dis-
tributions mentioned earlier.

Some other distributions have been suggested.
Scattering theory of course provides them in physics. Kendall
(1974) studied a model of bird navigation and derived the
(complicated) distribution of the angle where the bird first
hits its (circular) home territory — the Lack distribution.
In passing he discusses briefly the off-set or displaced normal
distribution which has arisen in other contexts. We will call it
the angular Gaussian in the next section. The angular Gaussian
has been suggested in two studies of the precision of palaeo-
magnetic measurements -- Harrison (1980), Briden and Arthur (1980),
the latter paper providing approximate statistical methods.

The mathematical and, particularly, the arithmetic relationships between the von Mises-Fisher, Brownian motion and angular Gaussian distributions is the subject of this paper.

2. The angular Gaussian

Fisher observed that his distribution may be derived from a Gaussian by conditioning. If X has a trivariate Gaussian distribution with mean vector μ and covariance matrix $\sigma^2 I$, set $R = |X|$, $L = X/R$, $\lambda = \mu/|\mu|$.

Then the joint density of R and L is

$$\frac{R^2}{(2\pi)^{3/2} \sigma^3} \exp\left\{- \frac{1}{2\sigma^2} (R^2 + |\mu|^2)\right\} \exp \frac{R|\mu|L'\lambda}{\sigma^2}$$

so the density of L on the unit sphere, conditional upon fixed R has the form (2) with $\kappa = R|\mu|/\sigma^2$, $\cos\theta = L'\lambda$. It is more surprising that he did not comment on the way palaeomagnetic directions are actually obtained. The three components of the magnetic intensity (let it be X) are measured by rotating the specimen in the same apparatus. Hence the above distribution for X is appropriate but the _unconditional_ distribution of L is what is required! Setting $S = R/\sigma$ and $m = |\mu|/\sigma$, this is

$$ag_3 (L,\lambda,m) = (\frac{1}{2\pi})^{3/2} \int_0^\infty S^2 \exp\left\{- \frac{1}{2} \left(S^2 + m^2 - 2SmL'\lambda\right)\right\} dS \quad (8)$$

It is a function only of $L'\lambda = \cos\theta$ and so rotationally symmetric about the mean direction λ. m is an a concentration parameter. The statistical aspects of (8), and its relation to palaeomagnetic work, are discussed in Watson (1980). Of course

if measurement error is not the only source of variability in the components of X, X may well have a general Gaussian distribution and distributions more complicated than (8) and (9) arise.

$$ag_2(L, \lambda, m) = \frac{1}{2\pi} \int_0^\infty S \exp\left\{- \frac{1}{2} (S^2 + m^2 - 2SmL'\lambda)\right\} dS \quad (9)$$

The densities (8) and (9) give the angular Gaussian distributions on the sphere and circle respectively.

The Fourier expansion for the von Mises density (1) is

$$vm(\theta, \kappa) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \frac{I_p(\kappa)}{I_0(\kappa)} \exp(ip\theta) \quad (10)$$

$$= \frac{1}{2\pi} \left(1 + 2 \sum_1^\infty \frac{I_p(\kappa)}{I_0(\kappa)} \cos(p\theta)\right)$$

since $I_p(\kappa) = I_{-p}(\kappa)$. Thus the Fourier expansion of $ag_2(L, \lambda, m)$ with $\cos \theta = L'\lambda$ is therefore

$$ag_2(L, \lambda, m) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{ip\theta} \int_0^\infty \exp\left\{- \frac{1}{2} (S^2 + m^2)\right\} S I_p(Sm) dS,$$

or

$$ag_2(\theta, m) = \frac{1}{2\pi} \left\{g_0(m) + 2 \sum_1^\infty g_p^{(2)}(m) \cos(p\theta)\right\}, \quad (11)$$

where

$$g_p^{(2)}(m) = \int_0^\infty S \frac{I_p(Sm)}{I_0(Sm)} \exp\left\{- \frac{1}{2} (S^2 + m^2)\right\} dS \quad (12)$$

Similarly, the spherical harmonic expansion of $ag_3(L, \lambda, m)$ or $ag_3(\theta, \phi, m)$ follows from the expansion of (2). This like (7) was given by Roberts and Ursell (1960) and is

$$= \sum_0^\infty \frac{2p + 1}{4\pi} \frac{I_{p + \frac{1}{2}}(\kappa)}{I_{\frac{1}{2}}(\kappa)} P_p(\cos\theta). \tag{13}$$

Hence the expansion of (8) is

$$a g_3\theta,\phi,m) = \sum_0^\infty \frac{2p + 1}{4\pi} g_p^{(3)}(m) P_p(\cos\theta) \tag{14}$$

where

$$g_p^{(3)}(m) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty S^2 \exp - \frac{1}{2}(S^2 + m^2) \frac{\sinh Sm}{Sm} \frac{I_{p + \frac{1}{2}}(Sm)}{I_{\frac{1}{2}}(Sm)} dS \tag{15}$$

The angular Gaussian is a special case of distributions obtained by integrating the length of a vector. Thus let f(z) be the density function of vectors z in $\mathbf{R}^q$ and set $z = \gamma \ell$ where γ is the length of z. Then the distribution of the direction ℓ on $\Omega\rho$, the surface of the unit sphere is $\mathbf{R}^q$, is

$$g(\ell) = \int_0^\infty \gamma^{q-1} f(\gamma\ell) d\gamma$$

This family of distributions is examined in Watson (1981).

3. Some mathematical relations between distributions

The classical way of matching two distributions with the same mean on the line is to adjust the parameters so they also have the same variance. It is easy mathematically but need not lead to a good probabilistic match. For directions the analogue is to match $E \cos \tilde{\theta}$ i.e. make the $p = 1$ terms in the Fourier (or spherical) expansions equal. Now that computing is easy, this or other methods may be used and compared. As we will see, it is not possible to improve much on $E \cos \theta$ matching for several definitions of a good match.

Roberts & Ursell (1960) began this work by showing that the Brownian distribution (7) and the Fisher distribution (2) matched well if one chose κ and V so that $E \cos \theta$ was equal for both. Comparing (7) and (13) this means they are related by

$$\frac{I_{3/2}(\kappa)}{I_{1/2}(\kappa)} = \exp(-V/2) \tag{16}$$

Since both are rotationally symmetric, the merit of the matching could be judged by $\sup| FI(\theta,\kappa)-BR_3(\theta,V)|$ where $FI(\theta,\kappa)$ and $BR_3(\theta,V)$ are cumulative distributions of θ easily obtained from (2) and (7) respectively. They found that then the supremum never exceeds (slightly over) .02 which it takes when $\exp-V/2 = 1/2$.

The author suggested to Stephens that he do the same in his thesis for the circular distributions (1) and (6). The matching relation is, comparing (6) and (10),

$$\frac{I_1(\kappa)}{I_0(\kappa)} = \exp(-V/2) \qquad (17)$$

Stephens (1963) found that the sup $|VM(\theta,\kappa) - BR_2(\theta,V)|$ never exceeded .0125 which was attained for the matched pair $\kappa = 1.4$, $V = 1.09$. Further, he found that the matching could hardly be improved beyond that implied by (17). The relationships (16) and (17) may be simplified when $\kappa \rightarrow 0$ and $\kappa \rightarrow \infty$.

These fascinating results led to a number of papers. Hartman and Watson (1974) showed that there is, in any number of dimensions, a distribution depending on κ and the dimensionality, for the diffusion time t so that the time mixture of the Brownian distribution is <u>exactly</u> equal to the von Mises-Fisher distribution. The Brownian distributions are infinitely divisible. Kent (1977, 1979, 1980) showed that this mixing distribution is infinitely divisible so that this is also true of the von Mises-Fisher distribution. J. Pitman (personal communication) has shown that it is not unique.

Kent (1978) showed that the von Mises(Fisher) distribution is the equilibrium distribution of the analogue of the Ornstein-Uhlenbeck process on the circle (sphere). The generator of this diffusion on the circle, e.g., is

$$G = \frac{1}{2} d^2/d\theta^2 - \lambda \sin\theta \, d/d\theta . \qquad (18)$$

Reuter (unpublished, Kent 1978) showed that the von Mises(Fisher) arises if a diffusion with drift starts at the origin and stops when it hits the unit circle (sphere).

Recently, Yor (1979) has given the explicit formula for a mixing distribution and a probabilistic explanation using two

dimensional Brownian motion. The formula was first obtained by
Kendall and appears in Kent's Ph.D. Thesis (1976) but was not
published. From (6) and (10), the von Mises will be a mixture
of Brownians if and only if for all integers p

$$\frac{I_p(\kappa)}{I_0(\kappa)} = \int_0^\infty e^{-p^2V/2} \, d\eta_\kappa(V),$$ (19)

where $\eta_\kappa(V)$ is a probability distribution. Yor and Kendall showed
that

$$d\eta_\kappa(V) = \frac{1}{\pi^{3/2} (2V)^{\frac{1}{2}}} \exp(\pi^2/2V) \frac{\kappa}{I_0(\kappa)} \psi_\kappa(V) dV$$ (20)

where

$$\psi_\kappa(V) = \int_0^\infty \exp\left\{-\left[(-x^2/2V)-\kappa\cosh x\right]\right\} \sinh x \, \sin\frac{\pi x}{V} dx. \; dx.$$ (20´)

Hartman and Watson (ibid) give a formula for obtaining from
(20) a mixing distribution in any number of dimensions.

The angular Gaussian is, like the von Mises-Fisher, a mixture
of Brownians. For the circle, the density is

$$ag_2 \; (\theta,m) = \frac{1}{2\pi} \left(1+2 \sum_1^\infty c_p(m)\cos p\theta\right) \quad ,$$ (21)

where

$$c_p = \int_0^\infty \exp\left(-\frac{1}{2}(s^2+m^2)\right) I_p(sm) s \, ds$$ (22)

Using the Kendall-Yor formula for $I_p(sm)$ and inverting the
order of integration, we see immediately that the mixing density
for the angular Gaussian is

$$\int_0^\infty \exp\left(-\tfrac{1}{2}(s^2+m^2)\right) \frac{d\eta}{du}sm(u)I_0(sm)sds \qquad (23)$$

It _is_ a density because

$$\exp\left(-\tfrac{1}{2}(s^2+m^2)\right) \ s \ I_0(ms) \qquad (24)$$

is a density -- for the length of a normal vector, in
fact. Similar results are easily obtained for ag_3.

Further, the angular Gaussian distributions may be infinitely
divisible, because the further mixing distributions (24) and
its higher dimension analogues are infinitely divisible. R^2 is
infinitely divisible but we have here mixed with R!

Returning to (18), we see that Kent's construction is of
a circular variant of the Orstein-Uhlenbeck process - the restoring
force towards the origin is proportional to $\sin\theta$, and his
equilibrium distribution is simply the Boltzman distribution since the
potential energy is proportional to $\cos\theta$. Formally, the equi-
librium density then satisfies

$$\frac{1}{2} \ \frac{d^2f}{d\theta^2} - \frac{d}{d\theta} \ (\lambda\sin\theta f) = 0 \qquad (25)$$

so that the density f is proportional to

$$\exp(2\lambda\cos\theta)$$

If, however, the restoring force is proportional to $\sin2\theta$, we get
attraction $(\lambda > o)$ to $\theta = \pi$ as well as $\theta = 0$, and the equilibrium
density is proportional to

$$\exp(\lambda\cos2\theta) \quad ,$$

which is the circular variant of (3). A more elaborate
construction gives the Bingham distribution (4).

4. Arithmetic and Graphical Comparisons of Distributions

The definitions of the von Mises-Fisher, Brownian and angular
Gaussian densities only suggest that they may be numerically similar
when their concentration parameters are either very small or very large.
If they can always be matched very closely by appropriate choices
of their concentration parameters, then for statistical analysis it
should matter little which is used even if a specific family is in-
dicated by stochastic modelling. We may then use the distribution
which is most convenient. For example it is much easier to simulate
the Brownian and angular Gaussian on the circle than the von Mises
distribution since they require only the generation of one and two
Gaussians respectively and simple algorithms. On the sphere, the angular
Gaussian and Fisher distributions are easier than the Brownian since
they require only three Gaussians and three uniforms, respectively, and
a simple algorithms. For statistical mathematics other "rankings"
obtain. Of course the distributions of some statistics calculated
from samples from different parents might be rather different even
if the parent populations are "well" matched i.e. could not be
distinguished with available data. Such statistics should not be used
since they are too sensitive to parental form. Hence the athmetical
comparisons of this section have many uses.

Kendall (1974) plotted sheaves of von Mises and Brownian densities and one $E\cos\theta$ matched pair of densities. In the discussion of his paper, John Kent showed graphs of $E\cos\theta$ matched triplets -- von Mises, Lack (derived from a bird model) and Brownian densities. The peaks at $\theta=o$ and in the tails are in that order, while in the middle of the range it reverses. The degree of agreement depends upon κ since the agreement becomes perfect as $\kappa \to o$ and $\kappa \to \infty$.

We wish to add the angular Gaussian to those comparisons, and to consider the spherical versions as well. Further $E\cos\theta$ matching may be supplemented by other methods. We also tried

(a) eye matching of densities (subjective)

(b) matching so that the supremum of the difference of the cumulative distributions of θ was a minimum (objective, obtained a computer search).

We begin with the circular versions. As mentioned earlier Stephens made the Brownian - von Mises $E\cos\theta$ comparisons.

To match the circular angular Gaussian to the von Mises by equating values of $E\cos\theta$, m and κ must be related by

$$\frac{I_1(\kappa)}{I_0(\kappa)} = \int_0^\infty \exp\left\{-\frac{1}{2}(s^2+m^2)\right\} I_1(sm)s\,ds. \qquad (26)$$

From asymptotic expansions, we have

$$\frac{I_1(\kappa)}{I_0(\kappa)} \sim \frac{\kappa}{2}(1 - \frac{\kappa^2}{8}), \kappa \to o,$$
$$\sim 1 - \frac{1}{2\kappa}, \kappa \to \infty.$$

To get limiting expressions for the right-hand side of (26) we may, for $m \to 0$, use

$$I_1(sm) = \sum_{r=0}^{\infty} \frac{1}{\Gamma(r+2)\Gamma(r+1)} \left(\frac{sm}{2}\right)^{2r+1},$$

and, for $m \to \infty$, use

$$I_1(sm) \sim (2\pi sm)^{-\frac{1}{2}}\{1 - \frac{3}{8sm}\} \exp sm.$$

Some calculations then yield the limiting forms of (26)

$$\kappa \sim m \left(\frac{\pi}{2}\right)^{\frac{1}{2}}, \quad \kappa \to 0,$$

$$\kappa \sim m^2, \quad \kappa \to \infty. \tag{27}$$

Criterion (a) suggests we look at the formulae for the densities. von Mises for $\kappa \to 0$ as approximately

$$\frac{1}{2\pi} (1+\kappa\cos\theta)$$

while the angular gaussian is, for $m \to 0$

$$\frac{1}{2\pi} \int_0^{\infty} s(1+ms\cos\theta)\exp(-s^2/2)ds,$$

$$= \frac{1}{2\pi} (1+ \frac{\sqrt{2\pi}}{2} m \cos\theta),$$

so we should set $\kappa = m (\pi/2)^{\frac{1}{2}}$ as in (27). When κ is large, θ is trivially shown to be Gaussian with variance κ^{-1}. When $m \to$ large, $\tan\theta = X_2/X_1$ where X_2 is Gaussian $(0,1)$ and X_1 is Gaussian $(m,1)$ and, since θ is small, $\tan \theta = \theta$. Hence θ is

asymptotically Gaussian $(0, m^{-2})$. Matching $\kappa = m^2$, again as in (27). We in fact found that "drawing" densities on a CRT and matching the AG(m fixed) and VM (κ to be found as $\kappa(m)$) densities by eye we got <u>very</u> similar $\kappa(m)$ to the values got by Ecosθ matching.

Figure 1 shows a sheaf of angular Gaussians. Figure 2 shows a sheaf of eye matched (Ecosθ match would be very similar) von Mises distributions. The relationships may be seen when transparencies of these figures are overlaid. When matched pairs are plotted together AG exceeds VM at $\theta=0$ and π and is less in the middle of the range. With Kent's results, mentioned above, we see that the angular Gaussian is the extreme member of the quadruplet -- but the density differences are small. It is in fact hard to appreciate such pictures (see Figures 3b, 4b, 5b) -- perhaps ratios of densities should be used.

Matching method (b) leads us to more revealing pictures of the differences of the cumulatives of the angular Gaussian, von Mises and Brownian distributions. For the first two distributions, the supremum matched κ is, for all m, slightly greater than the eye -- or Ecosθ -- matched κ. Of course the suprema obtained are smaller than for these latter methods. As is seen in Figure 3a, the supremum is .0125 the same as Stephens' value at $\kappa=1.4$, when the Brownian and von Mises are compared. In Figure 3b, the corresponding densities are drawn. Figure 4 illustrates the worst situation found (at $\kappa=2.646$, m=1.6) for the angular Gaussian and the von Mises -- the supremum is less than

in Figure 3a. In Figure 5, the worst case for Brownian and angular Gaussian is shown.

Turning now to the sphere, Roberts & Ursell did $E\cos\theta$ matching of the Fisher and Brownian distributions (the matching formula being (16) above) and looked also at the matching method (b). They found that the two methods give very similar results. Their conclusion was, of course, that it should be safe to use the Fisher distribution for statistics when the parental distribution was really Brownian.

Figure 6 gives a sheaf of Fisher densities for θ i.e., plots of

$$\frac{\kappa}{2\sinh\kappa} \exp(\kappa\cos\theta) \sin\theta \tag{28}$$

for $\kappa = 0(.25)1.5$, $2(1)10$. Figure 7 gives a sheaf of angular Gaussians densities of θ i.e., plots of

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty s^2 \exp\{-\tfrac{1}{2}(s^2+m^2-2sm\cos\theta)\}ds \, \sin\theta \tag{29}$$

for $m = 0$ $(.2)1$, $1.25(.25)$ 3. The matching is vivid only when transparencies of those figures are overlaid. The worst case we encountered is shown in Figure 8 where the difference of the cumulatives (Figure 8a) and the densities (Figure 8b) are shown. The supremum in Figure 8a is slightly less than .02 rather than slightly over .02 as Roberts & Ursell found. This occurred for a κ of 4.56 which corresponds to a reasonably disperse cluster. Our efforts at eye fitting of densities was much less successful

here. The supremum fitted kappas are shown in the brief table
below:

m	fitted κ
.1	.160
.3	.482
.5	.815
.75	1.26
1	1.74
1.5	2.94
3	9.4
4	16.4
5	25.4

These agree well with the formulae obtained from an E$\cos\theta$
match, or by mathematically matching the density functions (28)
and (29). By calculations similar to those used earlier, we find

$$\kappa \sim 4(2\pi)^{-\frac{1}{2}}m \qquad (\kappa \to 0),$$

$$\tag{30}$$

$$\kappa \sim m^2 \qquad (\kappa \to \infty).$$

Thus the angular Gaussian is as good a match as the Brownian
to the Fisher distribution, and to the von Mises.

LEGEND

FIGURE 1. A sheaf of angular Gaussian densities (formula 9)
for m=0(.2) 1.0, 1.25 (.25) 2.5 .

FIGURE 2. A sheaf of von Mises densities (formula 1) matched
by eye to the curves in Figure 1. Ecosθ matchings would be
very similar.

FIGURE 3. Three methods for assessing the best matching of the
Brownian and von Mises distributions when they are hardest
to match: v=1.06, κ=1.4 .
(a) A plot of the difference BR-VM of the cumulative
distributions.
(b) Plots of the br and vm densities.
(c) A plot of the relative error, (BR-VM)/VM .

FIGURE 4. Three methods for assessing the best matching of the
angular Gaussian and the von Mises distributions when they
are hardest to match: m=1.6, κ=2.646 .
(a) A plot of the difference AG-VM of the cumulative
distributions.
(b) Plots of the ag and vm densities.
(c) A plot of the relative error, (AG-VM)/VM .

FIGURE 5. Three methods for assessing the best Ecosθ matching
of the Brownian and angular Gaussian distributions when
they are hardest to match: m=1.2, v=0.818 .
(a) A plot of the difference AG-BR of the cumulative
distributions.
(b) Plots of the densities ag and br .
(c) A plot of the relative error, (AG-BR)/BR .

FIGURE 6. A sheaf of Fisher densities (formula 2) for θ for
κ=0(.25) 1.5, 2(1) 10.

FIGURE 7. A sheaf of angular Gaussian densities (formula 8) for
θ for m=0(.2) 1, 1.25 (.25) 3 .

FIGURE 8. Comparisons of the Fisher and angular Gaussian distribu-
tions when they are hardest to match: κ=4.56 , m=2 .
(a) A plot of the difference of the cumulatives for θ , AG-FI .
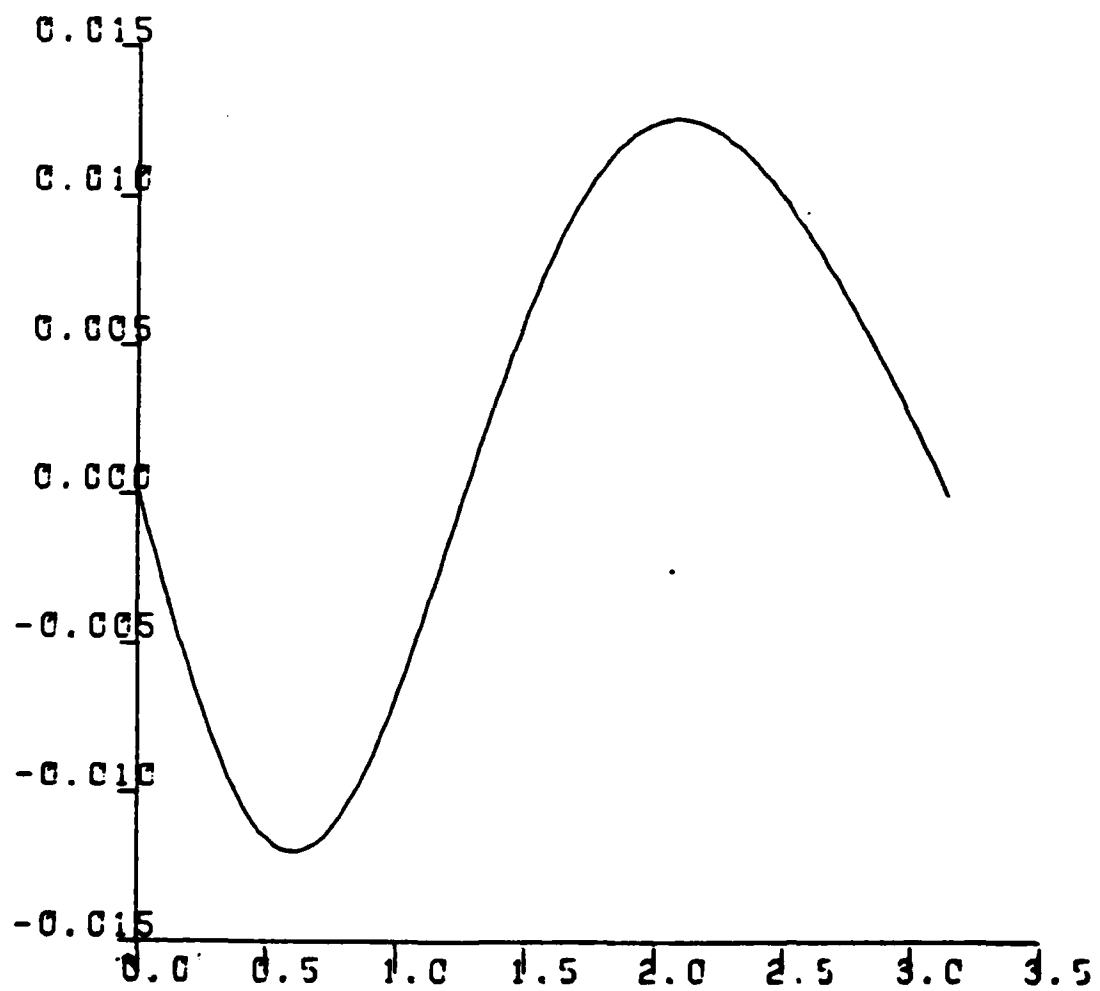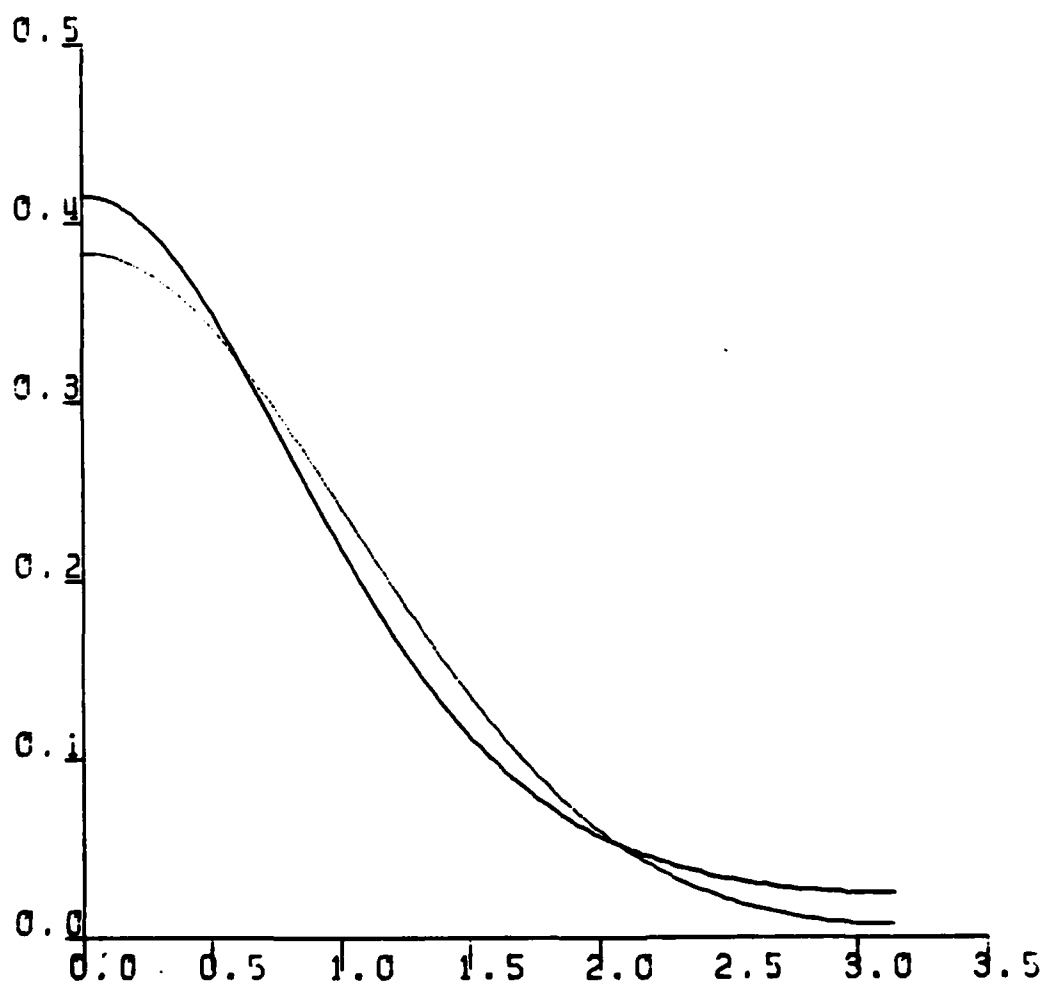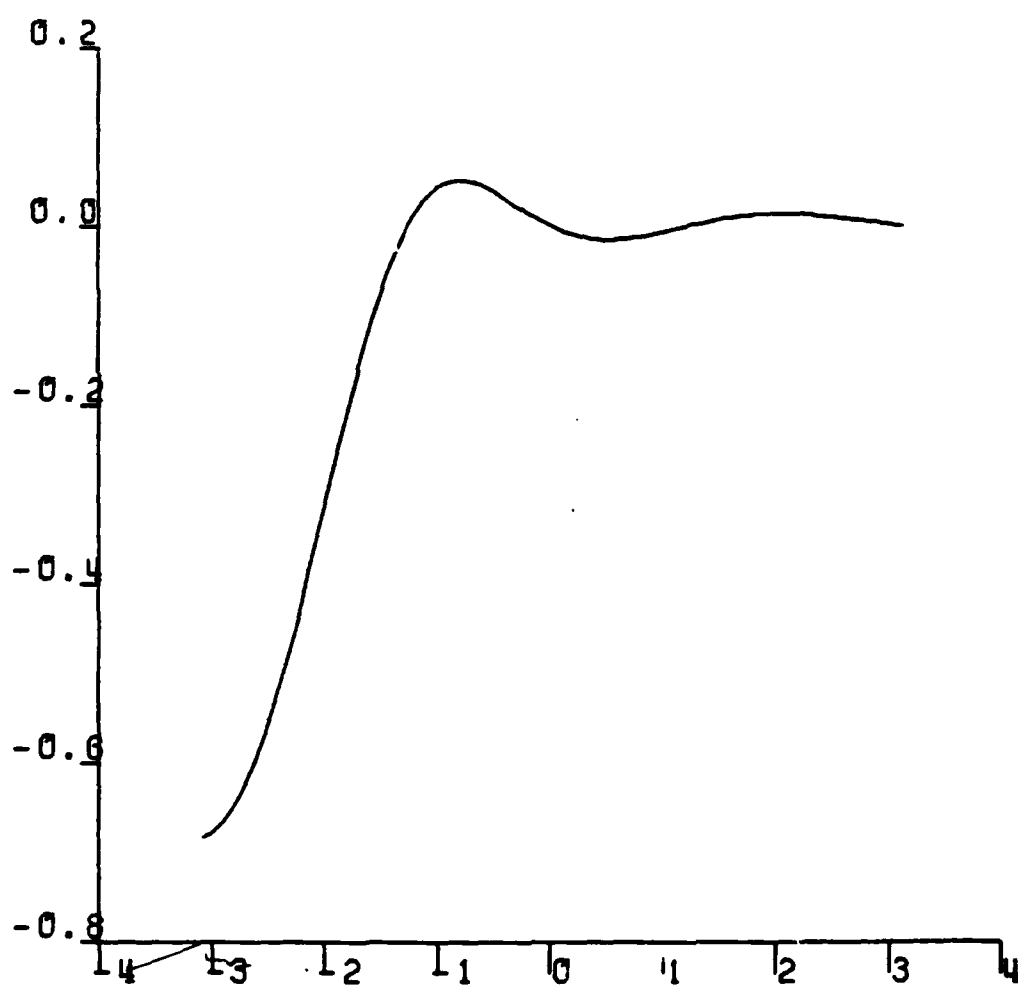(b) Plots of the two densities for θ , $ag_3(θ)$ and fi(θ) .
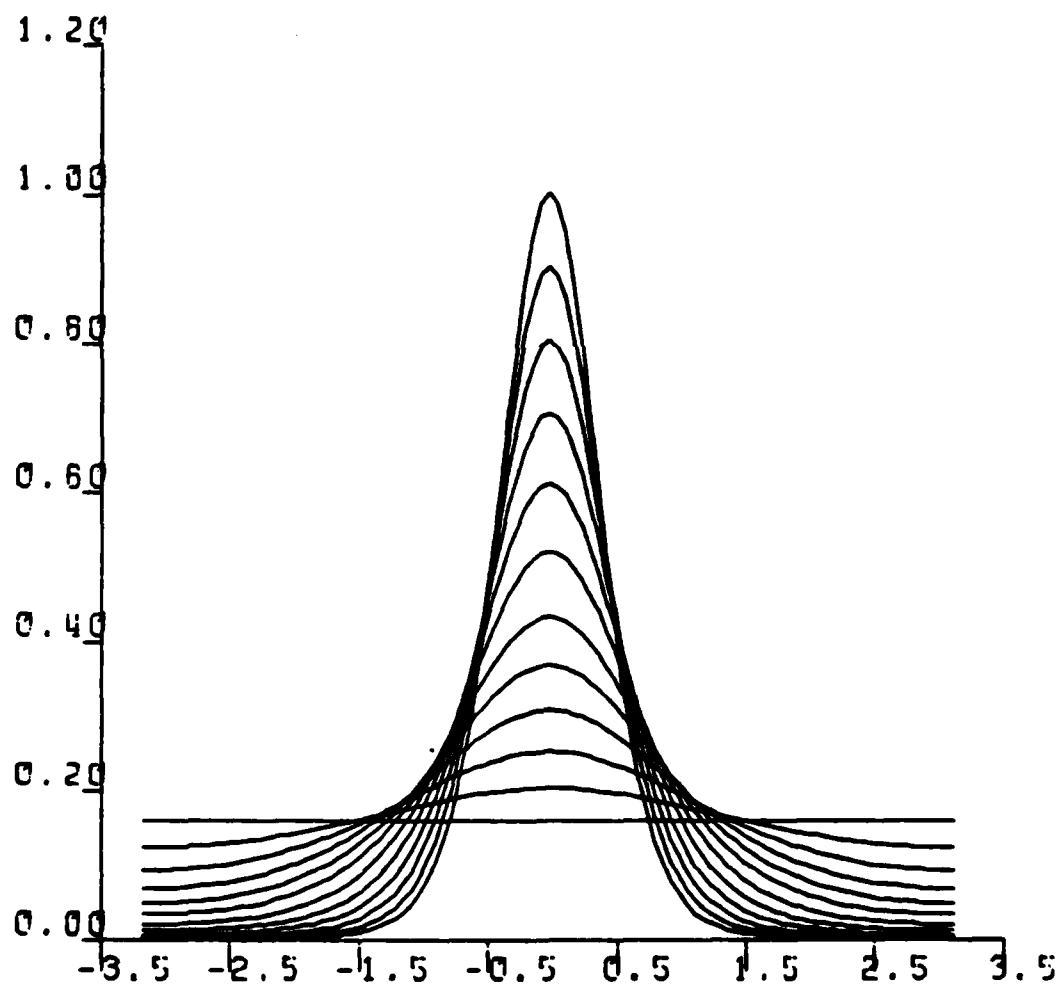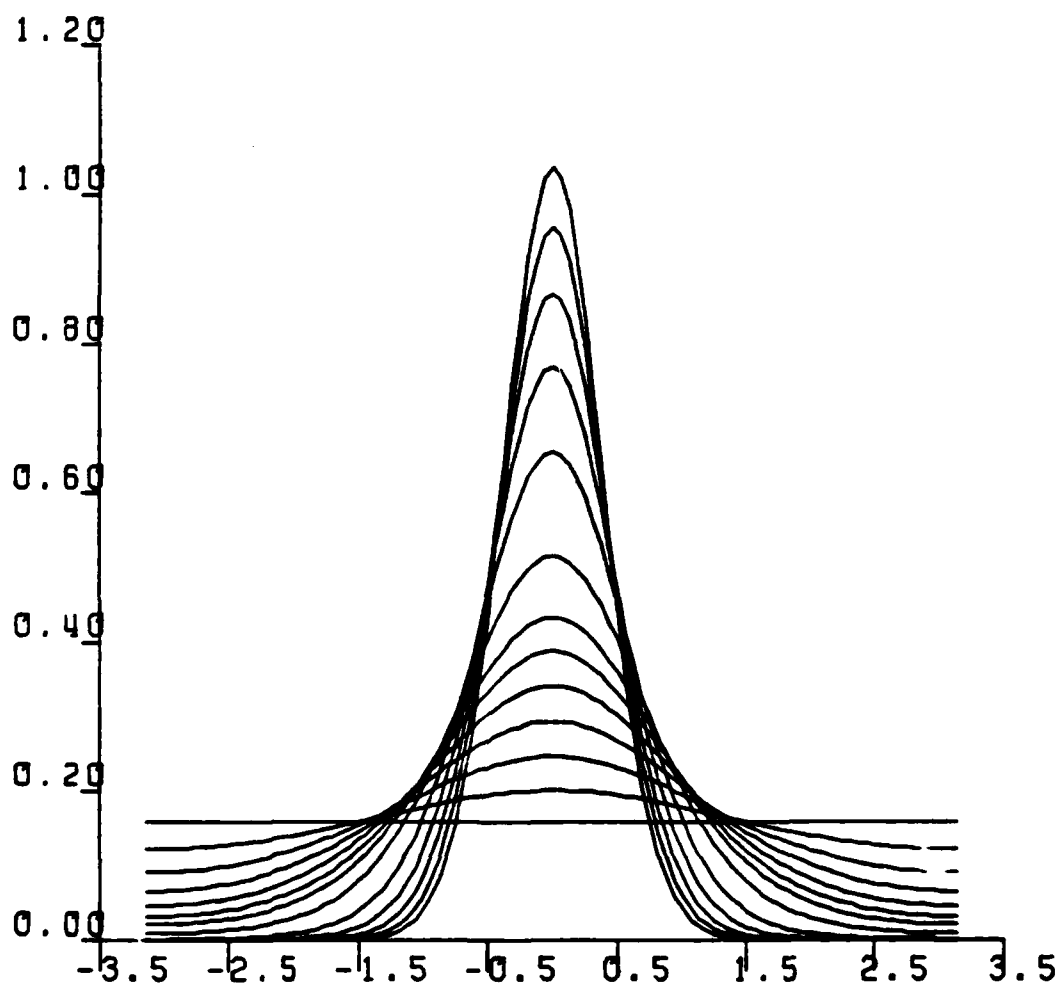
FIGURE 1.

FIGURE 2.

FIGURE 3a.

FIGURE 3b.

FIGURE 3c.
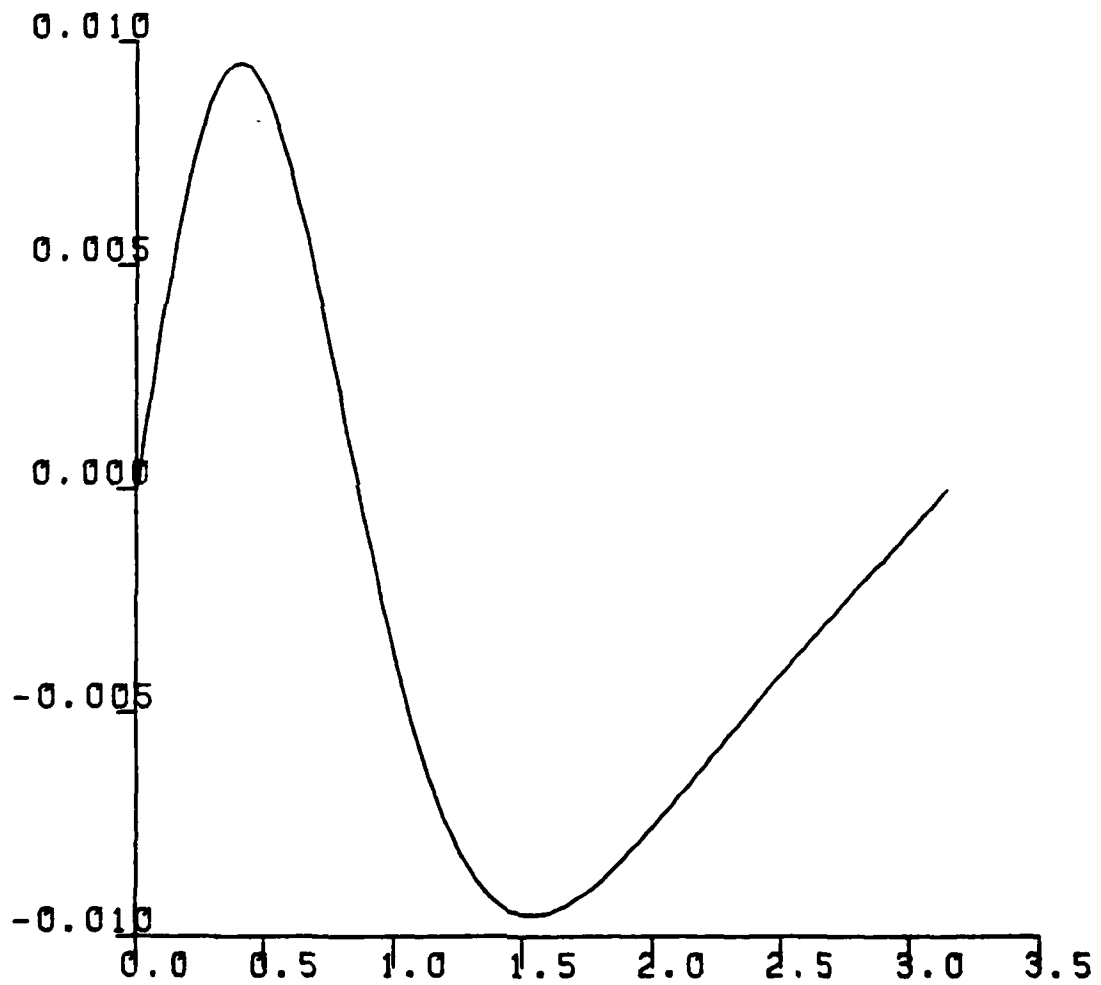
FIGURE 4b

FIGURE 4c.
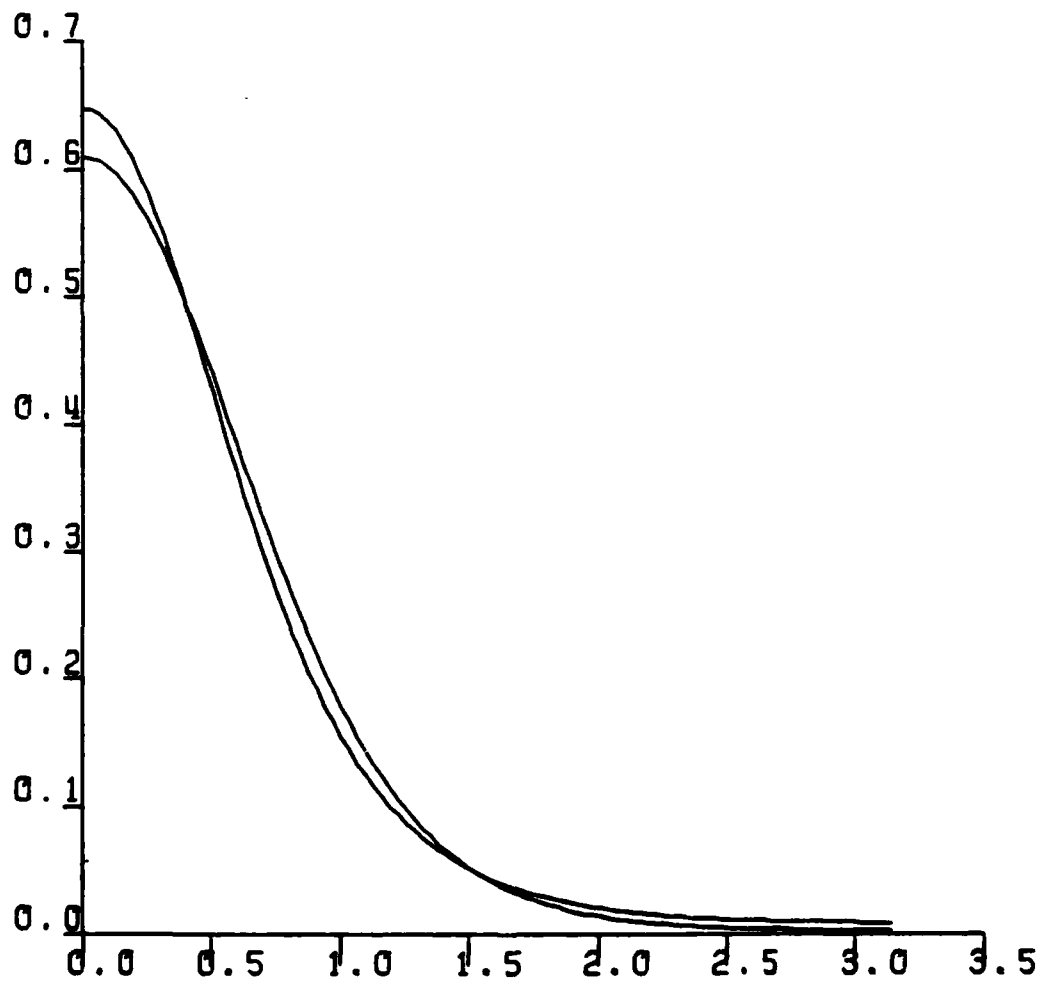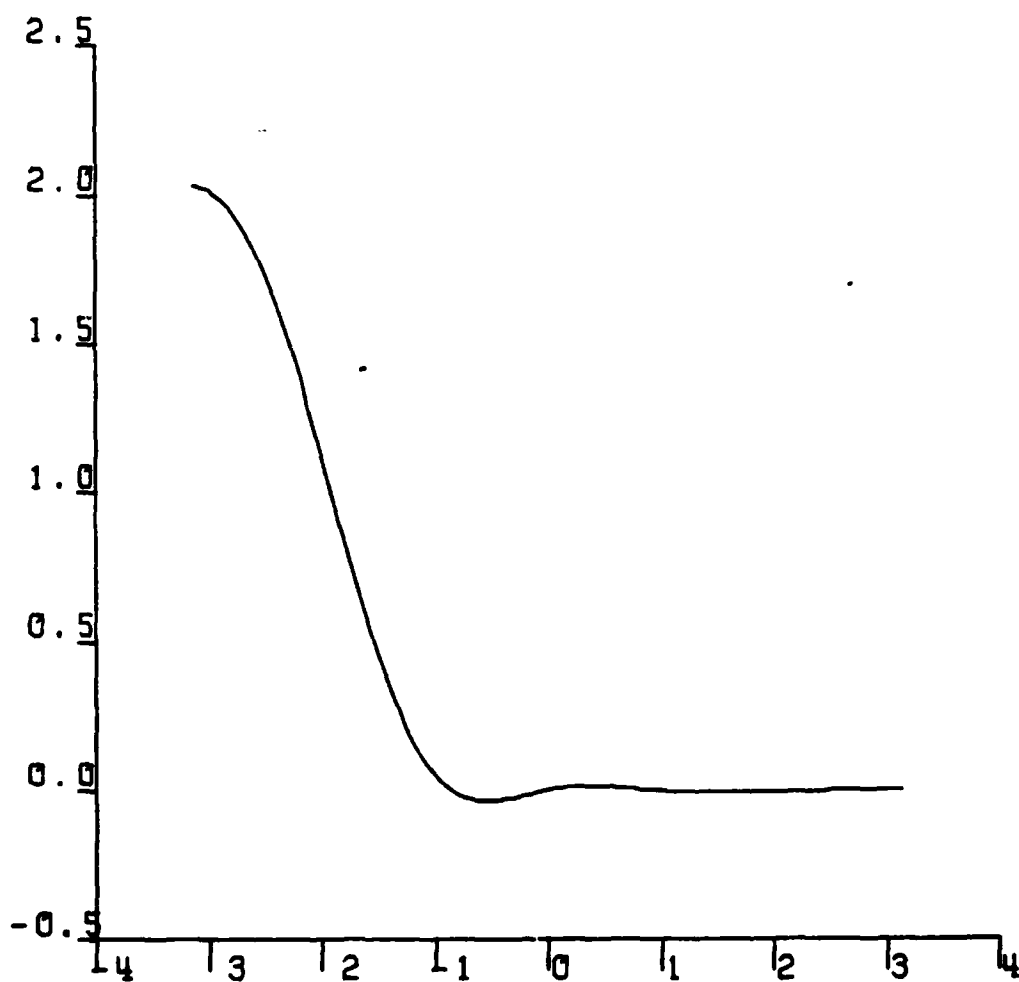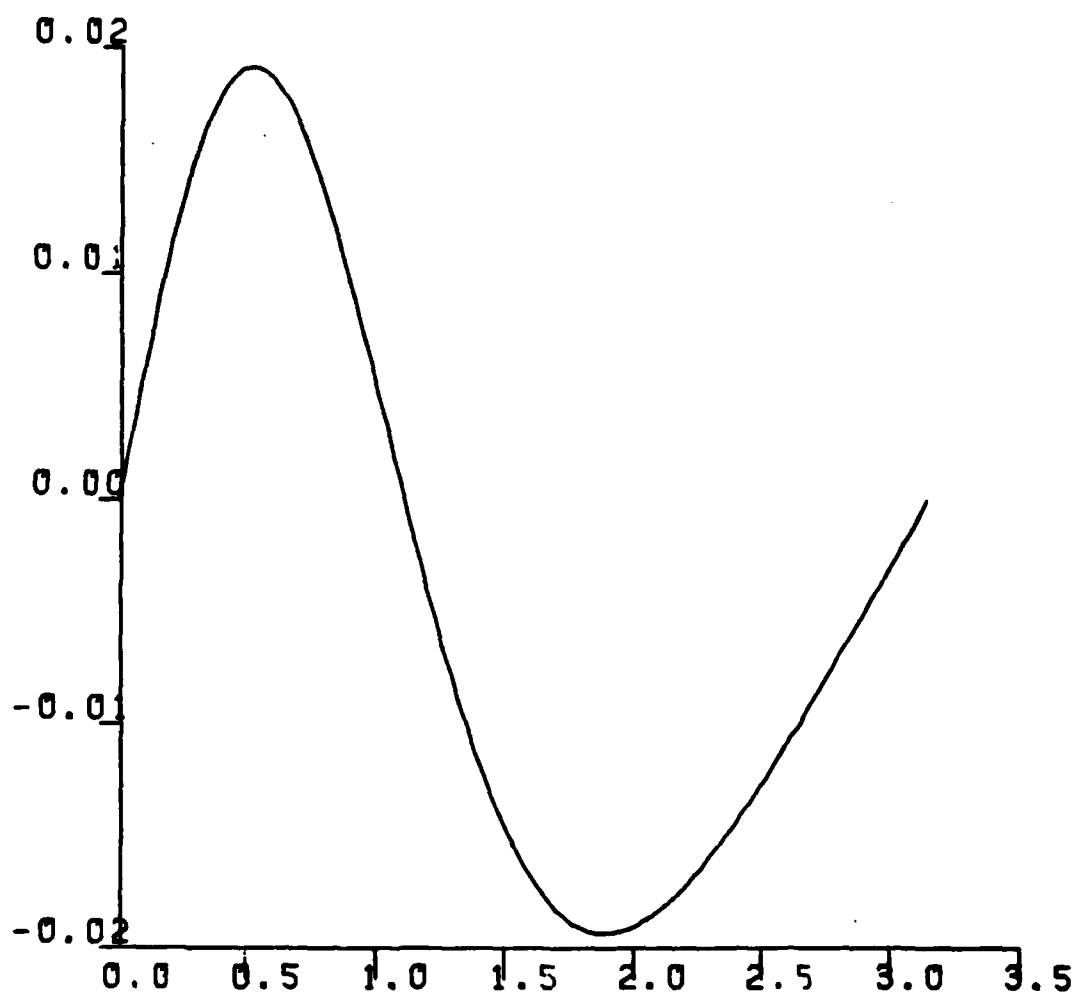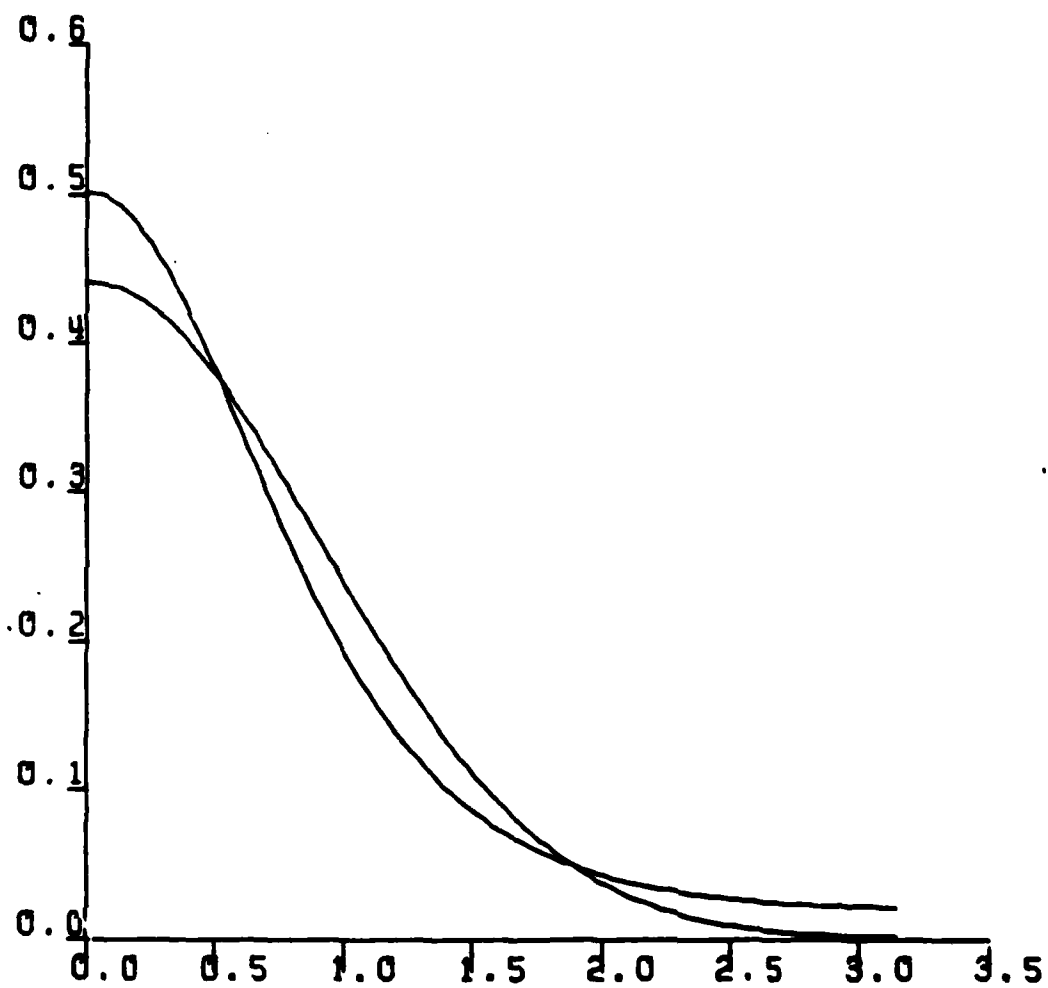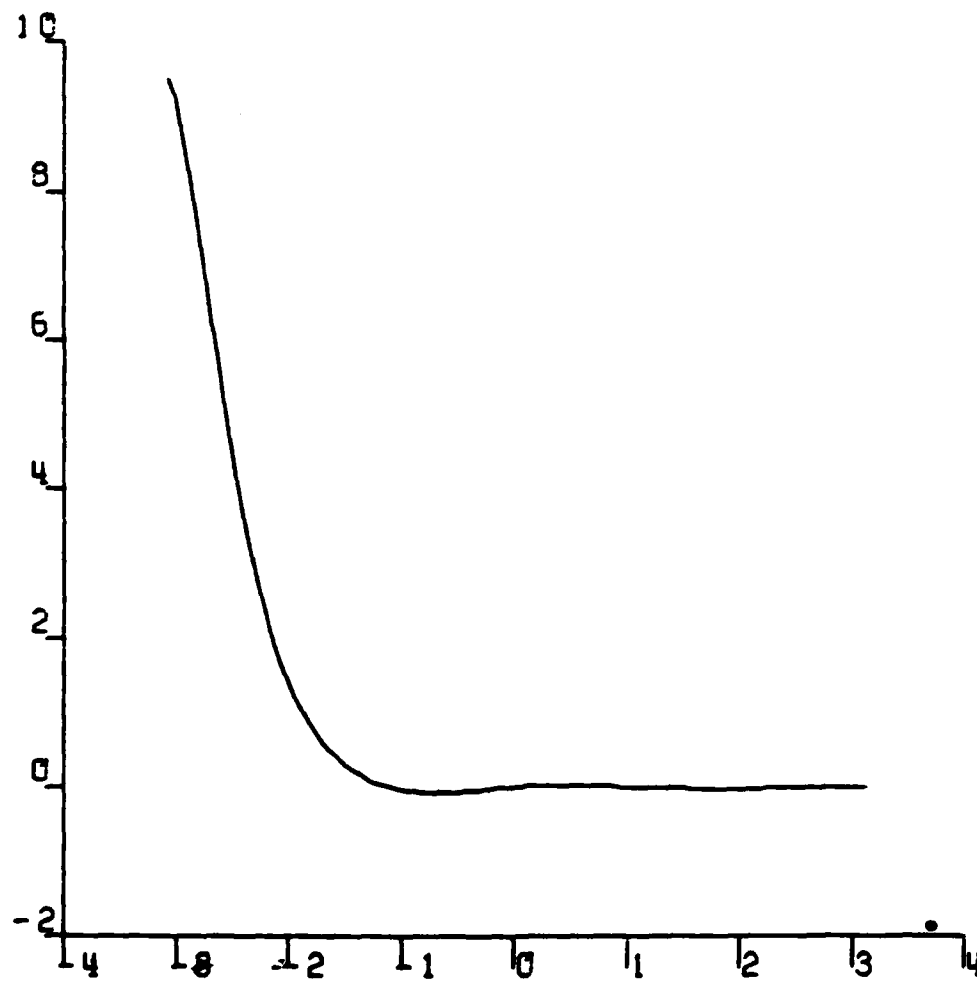
FIGURE 5a.

FIGURE 16
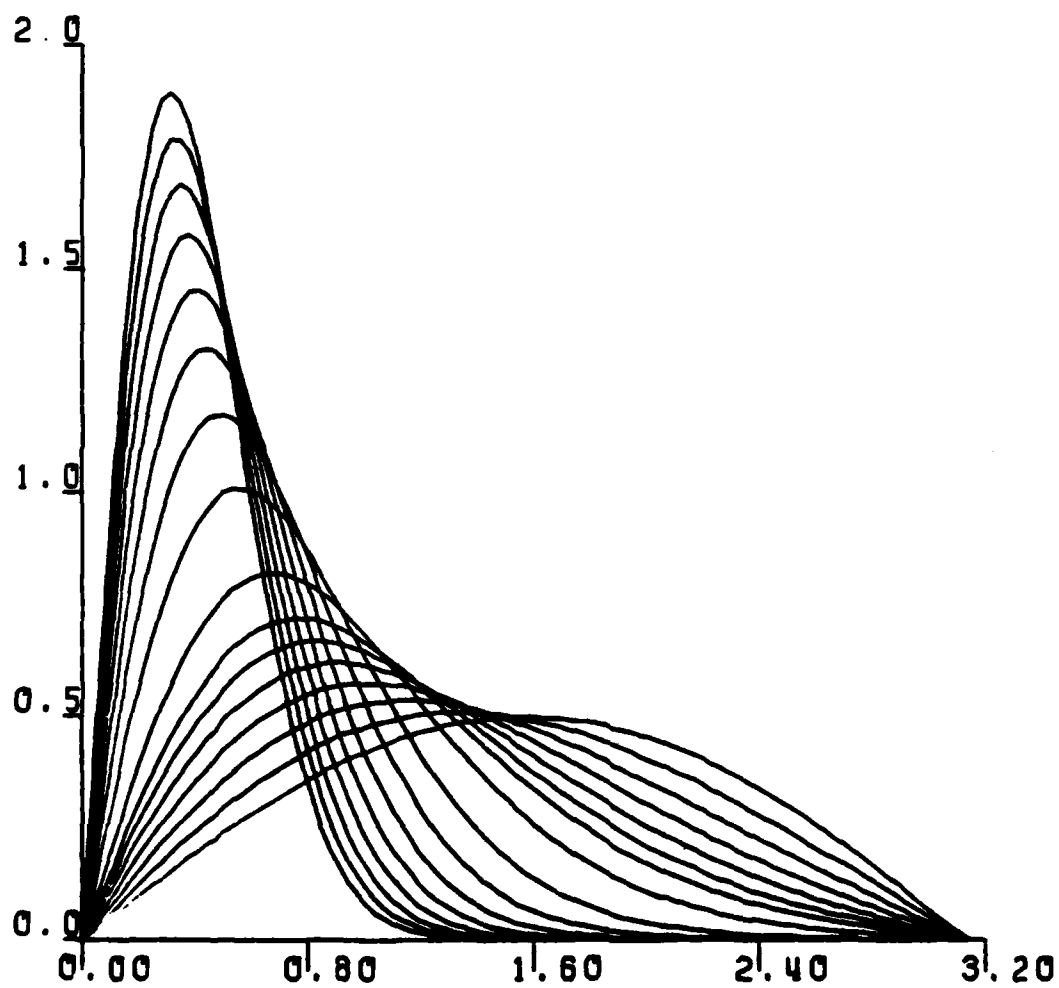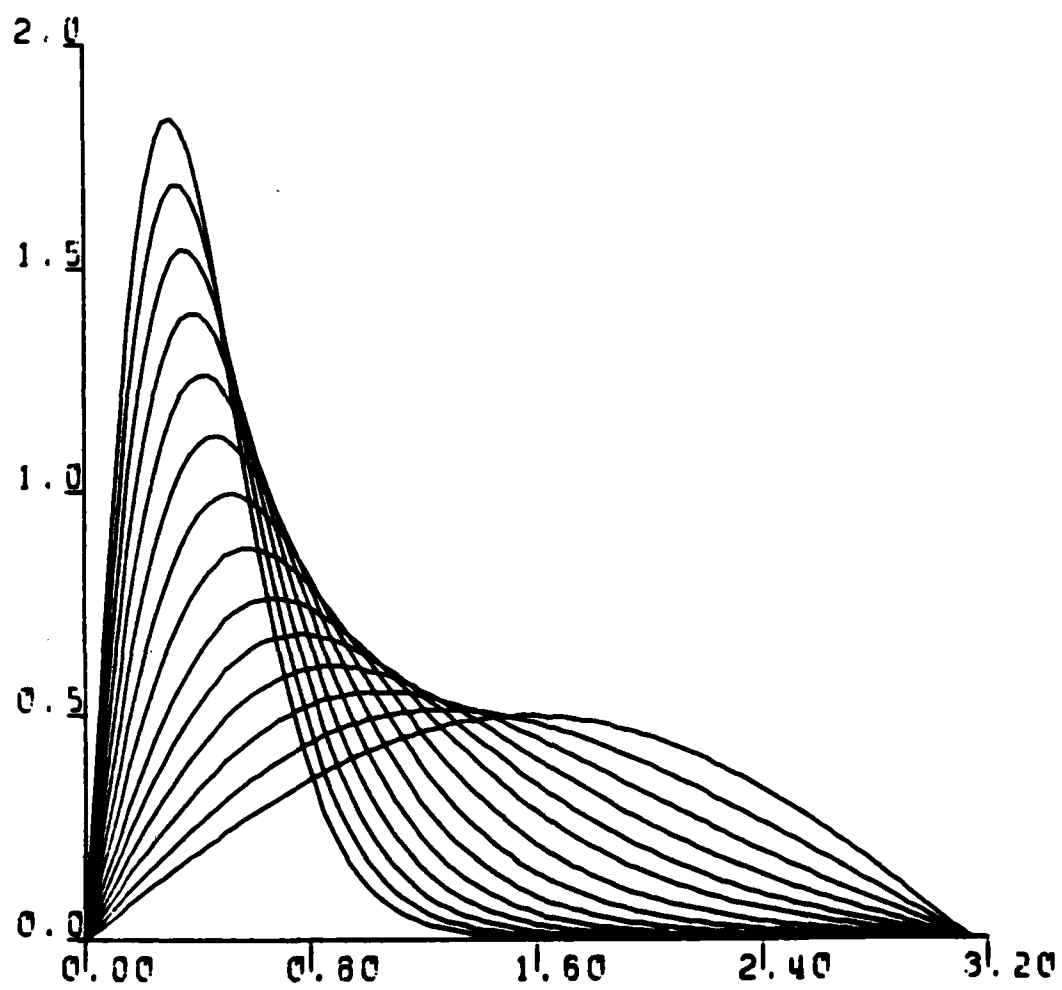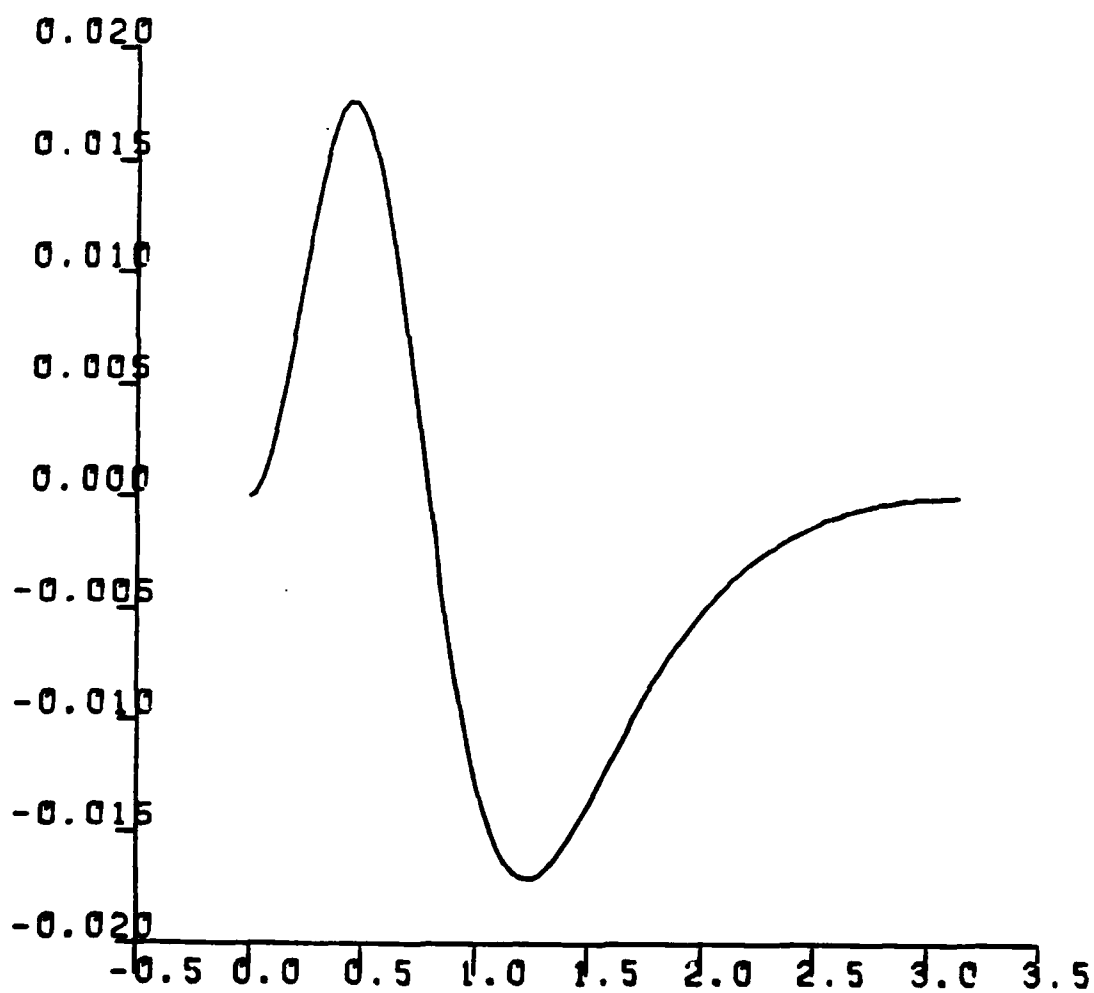
FIGURE 6.

FIGURE 7.

FIGURE 8a.
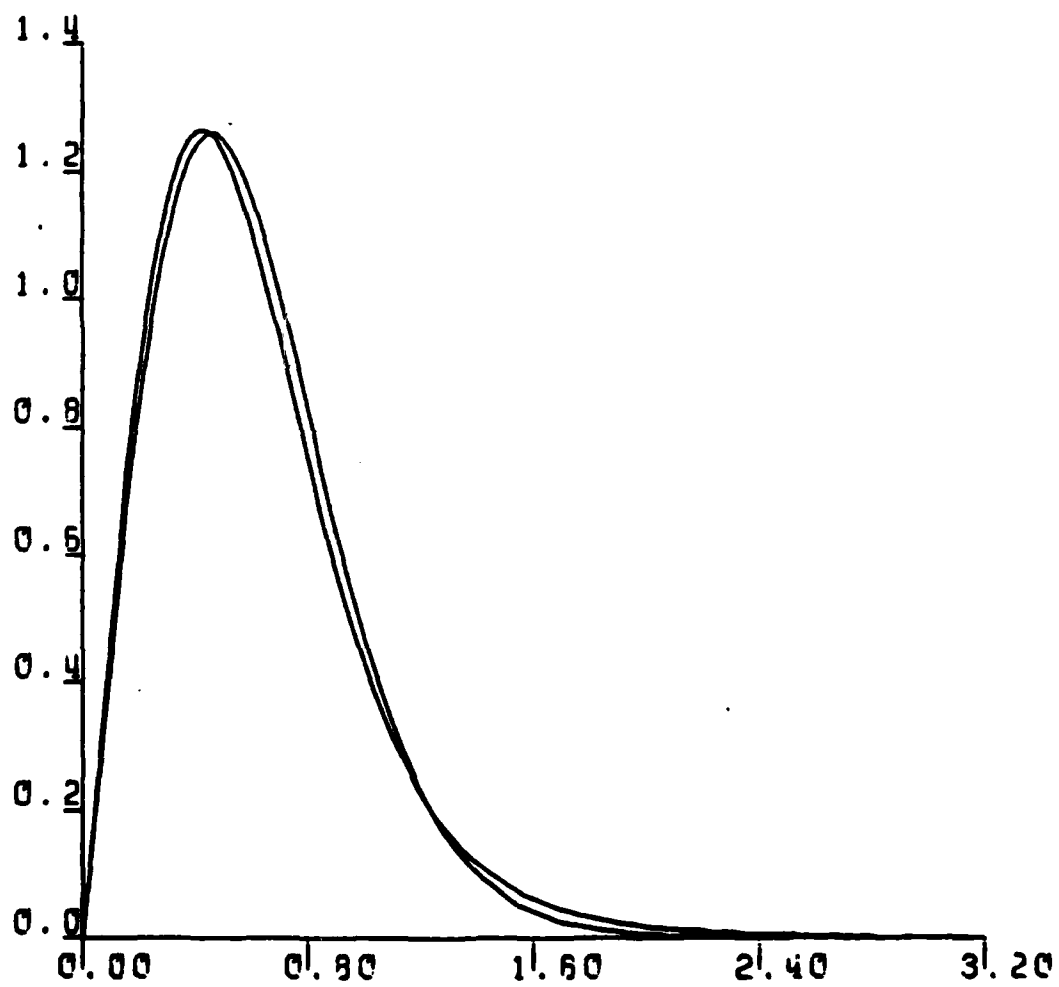
FIGURE 8b.

5. <u>Acknowledgements</u>

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A survey is made of the mathematical properties of, and the arithmetic relationships between various distributions on the circle and the sphere. Some new results are given. The Brownian motion and Angular Gaussian distributions are shown in computer drawn graphs to bracket the von Mises-Fisher distributions.

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